Section 9.7.4 Integrating a Vector-Valued Function

Recall: an antiderivative of a single-variable function $f(x)$ is a function $F(x)$ s.t.

$$F'(x) = f(x).$$

The indefinite integral $\int f(x)\,dx$ is the general antiderivative of f. From FTC

$$\int f(x)\,dx = F(x) + C.$$

Definition 9.7.5 An antiderivative of a vector-valued function $r(t)$ is a vector-valued function $R(t)$ such that

$$R'(t) = r(t).$$

The indefinite integral $\int r(t)\,dt$ is the general antiderivate of $r(t)$. ■

Since we differentiate componentwise, we also integrate component

$$\int r(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

If $R(t)$ is an antiderivative of $r(t)$

$$\int r(t)\,dt = R(t) + v$$

where v is a "constant" vector.

Activity 9.7.6

- Complete w/ your group.
- Class discussion.

(a) $a(t) = v'(t) = \left\langle -4\cos(2t), -2\sin(t), \frac{1}{(1+t)^2} \right\rangle$

(b) $r(t) = \int v(t)\,dt = \langle \cos(2t), 2\sin(t), t - \ln|1+t| \rangle + \langle a,b,c \rangle$

$\langle \frac{3}{2}, -1, 0 \rangle = r(0) = \langle 1, 0, 0 \rangle + \langle a,b,c \rangle$

So $\langle a,b,c \rangle = \langle \frac{1}{2}, -1, 0 \rangle$.

End of Section 9.7

Section 10.1 Limits

Reading Debrief

- Discuss 4-7 w/ your group.
- Questions?

• 5(a) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

Along the x-axis: $r(t) = \langle t, 0 \rangle$

$$\lim_{t\to 0} \frac{t\cdot 0^4}{t^2+0^8} = \lim_{t\to 0} 0 = 0.$$

Along the curve $x = y^4$: $r(t) = \langle t^4, t \rangle$

$$\lim_{t\to 0} \frac{t^4\cdot t^4}{t^8+t^8} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

Since limits along two different paths do not agree, the limit does not exist.

• 6(b) The function $f(x,y) = \frac{e^x + e^y}{e^{xy}-1}$ is continuous unless $e^{xy} - 1 = 0$.

Solve the eq: $e^{xy} = 1 \iff x\cdot y = 0$

$\iff x = 0$ or $y = 0$

Continuous everywhere except x and y axes.

Section 10.2 First-Order Partial Derivatives

Let $f(x,y)$ be a function and (a,b) a point. We can obtain a single-variable function

$$g(x) = f(x,b)$$

by setting $y=b$. The graph of $g(x)$ is the $y=b$ trace of f.

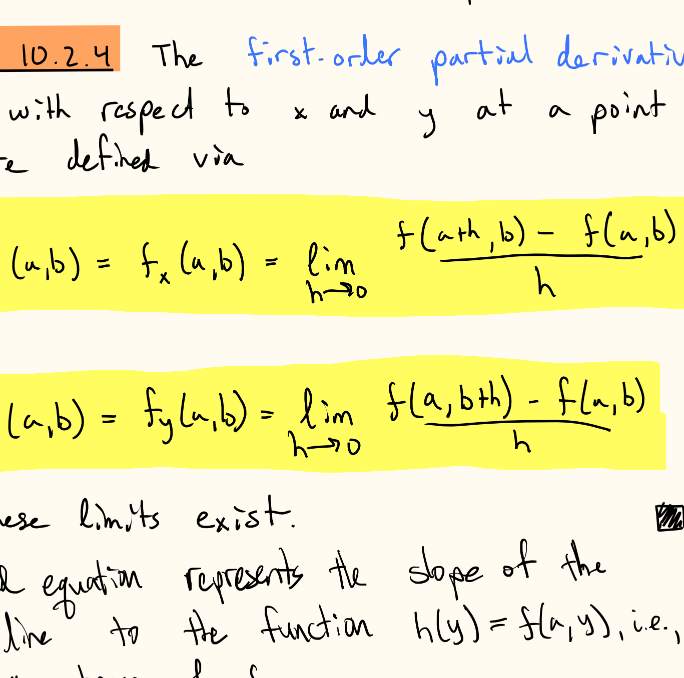
Since $g(x)$ is a single-variable function, we can take a derivative with respect to x:

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

This limit is therefore the slope of the tangent line to the $y=b$ trace of $f(x,y)$ at the point (a,b). Thus, it measures the rate of change of f "in the x-direction".

Tangent line to y=b trace of f



The slope of the green is equal to $g'(a)$. This is the partial derivative of f w/ respect to x.

Definition 10.2.4 The first-order partial derivatives of f with respect to x and y at a point (a,b) are defined via

- $$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
- $$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

when these limits exist. ■

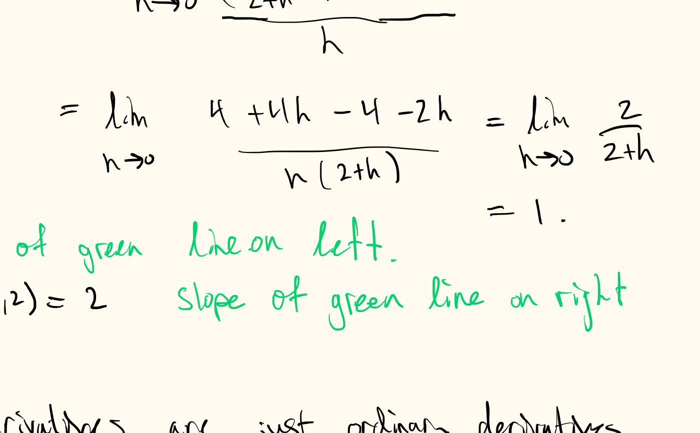
The second equation represents the slope of the tangent line to the function $h(y) = f(a,y)$, i.e., the $x=a$ trace of f.

Activity 10.2.2

- Complete Activity 10.2.2 and discuss w/ your group.
- Class discussion.

(a) and (c)

$f(x,y) = \frac{xy^2}{x+1}$



(b) $f_x(1,2) = \lim_{h\to 0} \frac{f(1+h,2) - f(1,2)}{h}$

$$= \lim_{h\to 0} \frac{\left(\frac{4(1+h)}{2+h}\right) - 2}{h}$$

$$= \lim_{h\to 0} \frac{4+4h-4-2h}{h(2+h)} = \lim_{h\to 0} \frac{2}{2+h}$$

$$= 1.$$

Slope of green line on left.

(d) $f_y(1,2) = 2$ Slope of green line on right

Partial derivatives are just ordinary derivatives of appropriate single-variable functions. Therefore all calc I differentiation rules apply.

Activity 10.2.3

- Complete together as a class.

(a) $f(x,y) = 3x^3 - 2x^2y^5$

To compute f_x: treat y as a constant, then differentiate as usual.

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(3x^3 - 2x^2y^5\right)$$

$$= 3\frac{\partial}{\partial x}x^3 - 2y^5\frac{\partial}{\partial x}x^2$$

$$= 3\cdot 3x^2 - 2y^5\cdot 2x$$

$$= 9x^2 - 4xy^5$$

To compute f_y: treat x as a constant.

$$f_y(x,y) = -10x^2y^4$$

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

(b) $f(x,y) = \frac{xy^2}{x+1}$ $f_x(x,y) = \frac{y^2(x+1) - xy^2}{(x+1)^2}$

$$= \frac{y^2}{(x+1)^2}$$

$$f_y(x,y) = \frac{2xy}{x+1}$$

(c) $g(r,s) = rs\cos(r)$ $g_r = s(\cos(r) - r\sin(r))$

$g_s = r\cos(r)$

(d)-(e) For partial derivatives of a function of three or more variables, treat all other variables as constant.

Section 10.2.2 Interpretations of First-Order Partials

Activity 10.2.4

- Complete w/ your group
- Class discussion.

(a) look at the limit definition:

$$C_T = \lim_{h\to 0} \frac{C(T+h,S,D) - C(T,S,D)}{h}$$

units: $(m/s)/C$

Instantaneous rate of change of speed of sound w.r.t temperature at a fixed salinity and depth.

$$C_S = \lim_{h\to 0} \frac{C(T,S+h,D) - C(T,S,D)}{h}$$

units: $\frac{m/s}{g/L} = \frac{m\cdot L}{g\cdot s}$

The instantaneous rate of change of speed of sound w.r.t. Salinity at a fixed temp and depth.

$$C_D = \lim_{h\to 0} \frac{C(T,S,D+h) - C(T,S,D)}{h}$$

units: $m/s/m = \frac{1}{s} = hz$

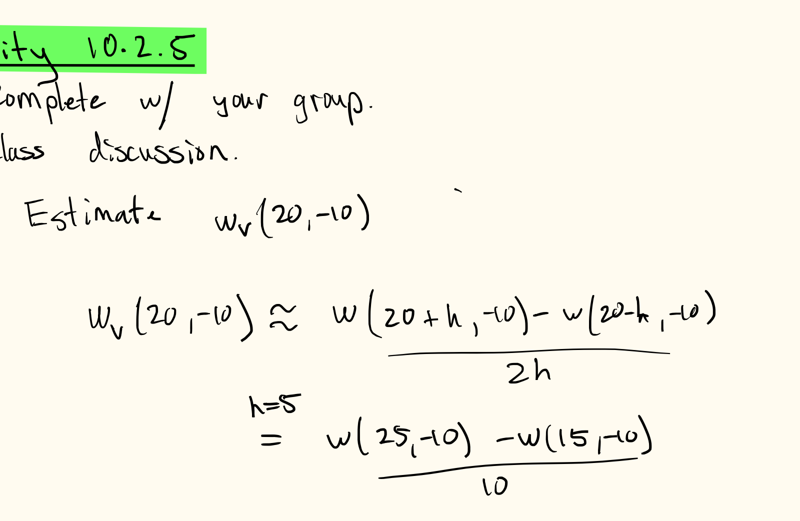
Frequency of sound wave at a fixed temp and salinity

Section 10.2.3 Estimating Partial Derivatives

The derivative $f'(a)$ of a single-variable function at $x=a$ can be estimated using the symmetric difference:

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

where h is a small number.



We can apply the same formula to a trace in order to estimate partial derivatives.

Activity 10.2.5

- Complete w/ your group.
- Class discussion.

(a) Estimate $w_v(20,-10)$

$$w_v(20,-10) \approx \frac{w(20+h,-10) - w(20-h,-10)}{2h}$$

$$\overset{h=5}{=} \frac{w(25,-10) - w(15,-10)}{10}$$

$$= \frac{-37 - (-32)}{10} = -\frac{1}{2}\ {}^\circ F/m/h$$

You will feel $\frac{1}{2}{}^\circ F$ colder for every 1mph increase in wind speed when the temperature is $-10^\circ F$.

(b) $w_T(20,-10) \approx \frac{w(20,-10+5) - w(20,-10-5)}{2\cdot 5}$

$$= \frac{-29 - (-42)}{10} = \frac{13}{10} = 1.3\ {}^\circ F/{}^\circ F$$

No units?

You feel $1.3\ {}^\circ F$ colder for every $1^\circ F$ change in temp when wind is going 20mph.